

Random states for robust quantum metrology

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... “[that] aims to use strong quantum correlations in order to develop systems, involving large-scale entanglement, that outperform classical systems in a series of relevant applications.”

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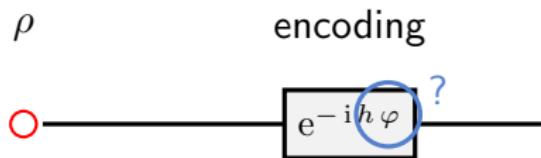
— <http://qurope.eu/projects/siqs>

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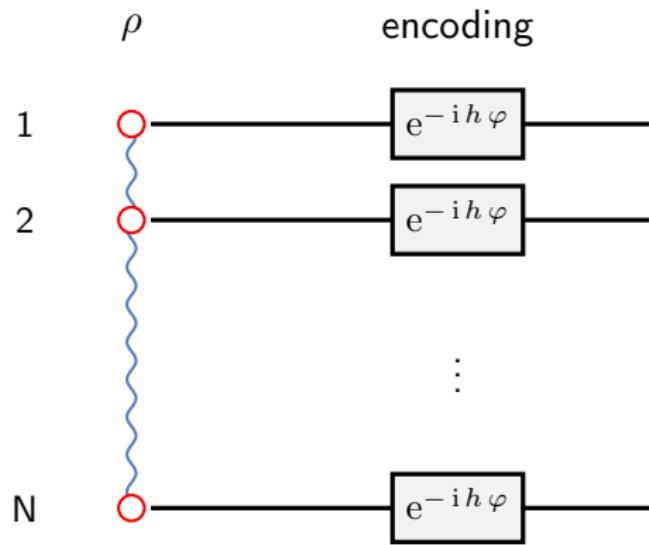
Phase estimation

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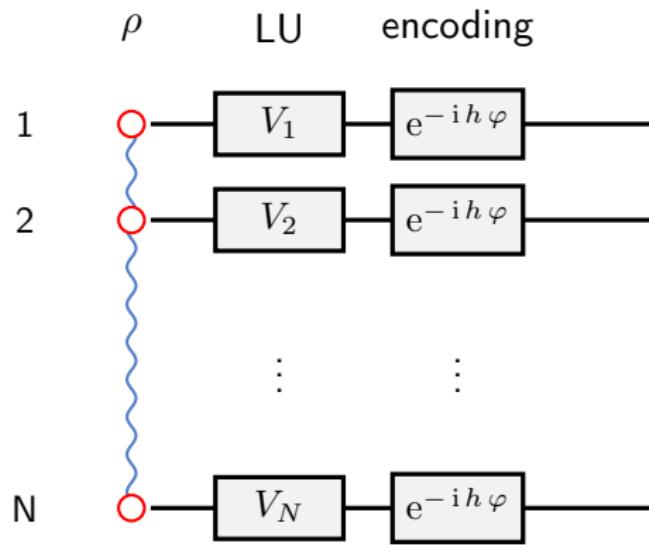
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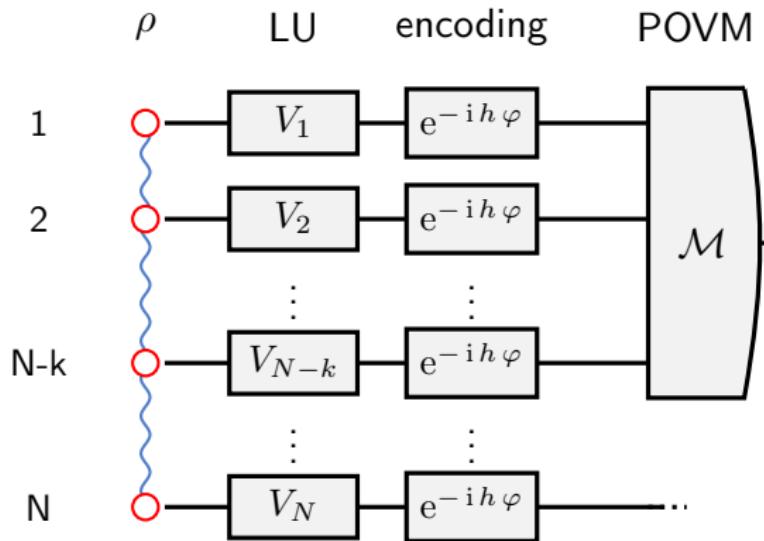
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Phase estimation (with local unitary optimization)



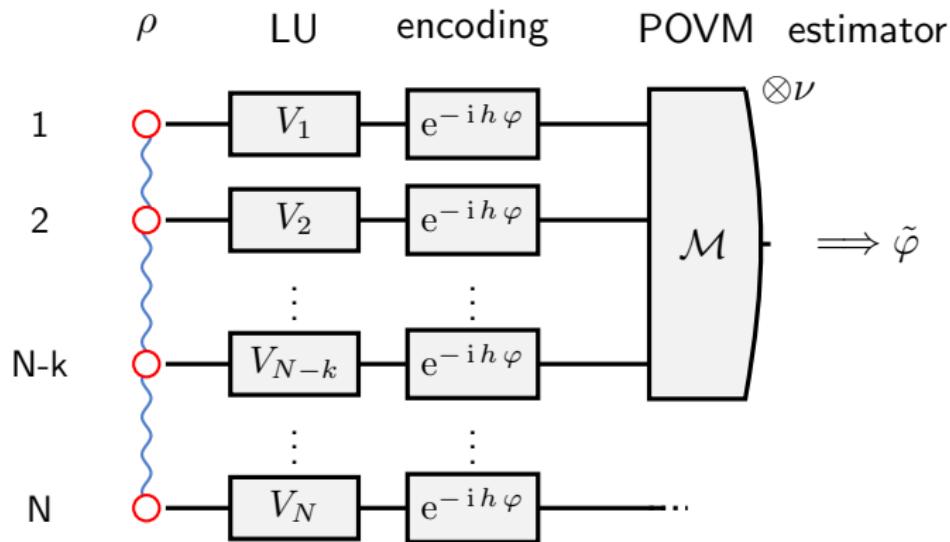
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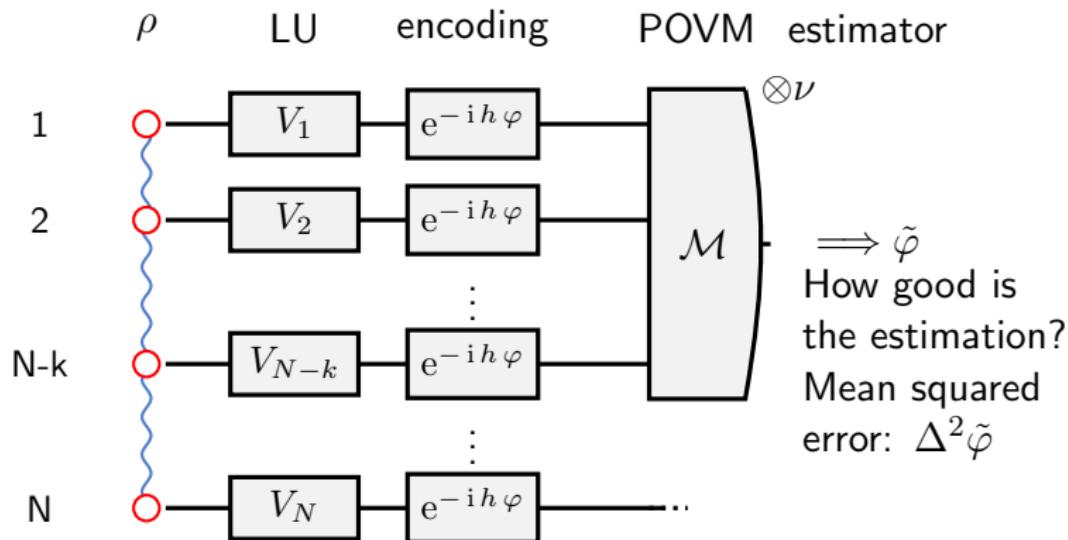
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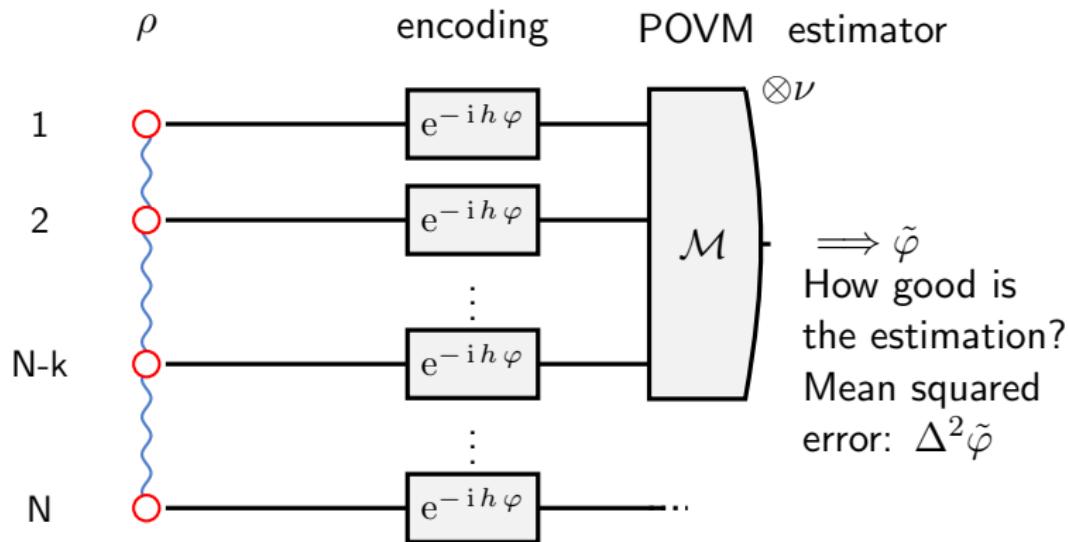
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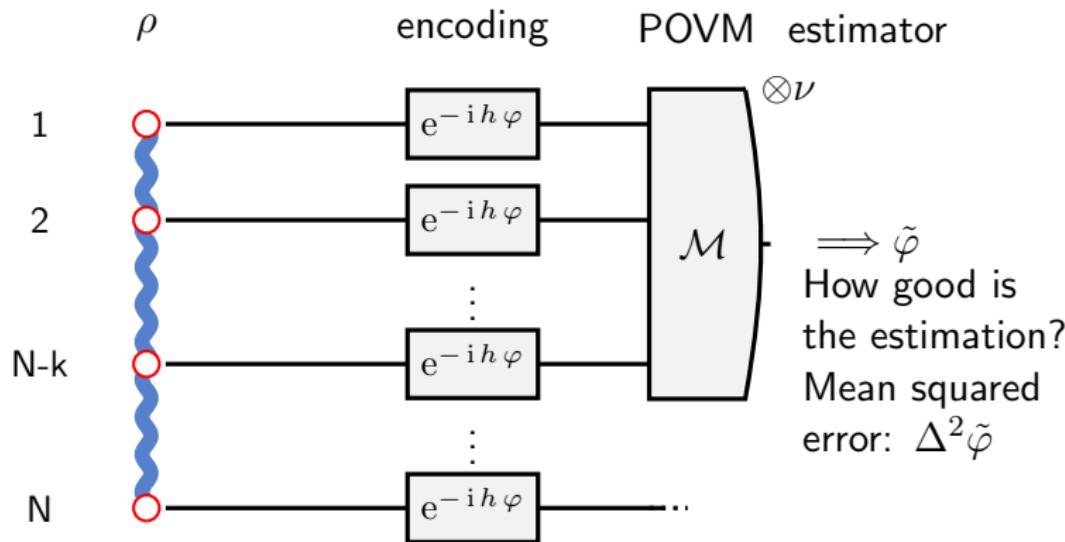
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Cramer Rao bound

Cramér-Rao Bound [3, 4] for unbiased estimators:

$$\Delta^2 \tilde{\varphi} \geq \frac{1}{\nu F_{\text{cl}}(\{p_{n|\varphi}\})}$$

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Moreover: Tight in the limit of many repetitions $\nu \rightarrow \infty$ [5].

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LU optimization:

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$$F^{\text{LU}} (\rho, H) := \sup_{V=V^{(1)} \otimes \dots \otimes V^{(N)}} F(V \rho V^\dagger, H)$$

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Questions:

- How generic is usefulness?
- Which types of states are useful?
- And for which measurements?
- How robust is the usefulness?
- What is the role of entanglement?

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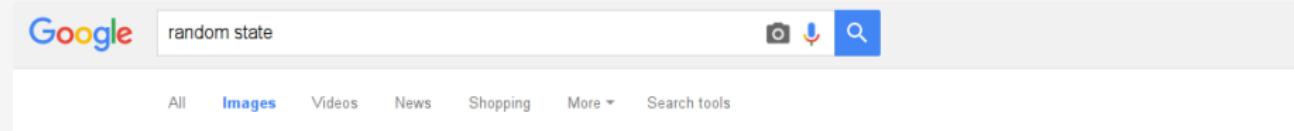
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Random states

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The United States of Shame

What is your state the worst at?



Google autocomplete results:

“Why is [state] so...”



as of January 2014

ISLAMIC TERRORIST NETWORK IN AMERICA



THE MOST FAMOUS BRAND FROM EACH STATE



Random states

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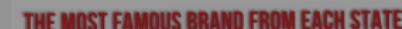
Random states

The Early summary of results

- Random states of distinguishable particles **not useful**.
- Random symmetric states (bosons) **typically are useful**.

This slide is a screenshot of a Google search results page for the query "random state". The search bar at the top shows the query. Below the search bar, the "Images" tab is selected, while other tabs like "All", "Videos", "News", "Shopping", and "More" are visible. The main content area displays a slide titled "The Early summary of results" with two bullet points. The slide is framed by a white border. The background of the slide features several small, overlapping images: a map of the United States with various labels like "Bestiarity", "Homeless population", "Crime", "Air pollution", and "Cost of Living"; a logo for "pleated jeans"; a map of the world with labels like "expensive", "affordable", "modest", and "cheap"; and a small inset map of the United States with labels like "white", "black", "Asian", "Hispanic", "Native American", "American Indian", "Alaskan Native", "Asian/Pacific Islander", "Asian American", "Hispanic/Latino", "White", "Black or African American", "American Indian and Alaskan Native", "Asian/Pacific Islander", "Two or more races", and "Other". At the bottom right, there is a timestamp "as of January 2014".

as of January 2014



Random states

The following slide is titled "Early summary of results" and contains three bullet points:

- Random states of distinguishable particles **not useful**.
- Random **symmetric states** (bosons) **typically are useful**.
- Exists one **fixed measurement** to access their usefulness.

The slide is framed by a white border and features a small map of the United States in the background. The map includes labels such as "Bestiarity" (red), "Homeless population" (yellow), "Crime" (blue), and "Air Pollution" (green). A small inset map of Canada is also visible in the top right corner, with labels like "white", "expensive", "poor", "isolated", "bad", "scoring", and "rich". At the bottom of the slide, there are several small logos and text snippets, including "Cost of Living" (red), "pleated jeans" (blue), and "Produced by the Complexity Research Lab".

as of January 2014



Random states

Early summary of results

- Random states of distinguishable particles **not useful**.
- Random **symmetric states** (bosons) **typically are useful**.
- Exists one **fixed measurement** to access their usefulness.
- Are **robust** against depolarization and particle loss.

as of January 2014

ISLAMIC TERRORIST NETWORK IN AMERICA

THE MOST FAMOUS BRAND FROM EACH STATE

Random states

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Early summary of results

- Random states of distinguishable particles **not useful**.
- Random **symmetric states** (bosons) **typically are useful**.
- Exists one **fixed measurement** to access their usefulness.
- Are **robust** against depolarization and particle loss.
- They can be generated with **random optical circuits**.

Random states more seriously

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$\mu(\mathcal{H})$ is the unique uniform measure on $SU(\mathcal{H})$

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- Random isospectral states:

Fix density matrix σ on some Hilbert space \mathcal{H} , then

$$U\sigma U^\dagger \quad \text{with} \quad U \sim \mu(\mathcal{H})$$

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We will consider:

d -level spins

$$\mathcal{H}_N := (\mathbb{C}^d)^{\otimes N}$$

symmetric (bosonic) subspace

$$\mathcal{S}_N := \text{span}\{|\psi\rangle^{\otimes N} : |\psi\rangle \in \mathbb{C}^d\}$$

Measure concentration

Levy's lemma

Let $f : \mathrm{SU}(\mathcal{H}) \rightarrow \mathbb{R}$ be L -Lipschitz, then

$$\Pr_{U \sim \mu(\mathcal{H})} \left(\left| f(U) - \mathbb{E}_{U \sim \mu(\mathcal{H})} f \right| \geq \epsilon \right) \leq 2 \exp \left(- \frac{|\mathcal{H}| \epsilon^2}{4 L^2} \right).$$

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We apply this to

$$F(U) := F(U \sigma U^\dagger, H) \quad \text{with } \sigma, H \text{ fixed}$$

and similar functions F^{LU} , F_{cl} , ...

Typical QFI of random states

Typical QFI of distinguishable particles

Random states of **distinguishable particles** are **not useful** for metrology:

Theorem

$$\Pr(\text{random state is useful}) \leq \exp(-\text{large in } N)$$

Typical QFI of distinguishable particles

Random states of **distinguishable particles** are **not useful for metrology**:

Theorem

$$\Pr_{U \sim \mu(\mathcal{H}_N)} \left(F^{\text{LU}}(U) \notin \mathcal{O}(N) \right) \leq \exp \left(-\Theta \left(\frac{|\mathcal{H}_N|}{N^2} \right) \right)$$

What about symmetric states?

Typical QFI of symmetric states

Random (bosonic) symmetric states are useful for metrology:

Theorem

$$\Pr_{U \sim \mu(\mathcal{S}_N)} (F(U) \notin \Theta(\delta^2 N^2)) \leq \exp(-\Theta(\delta^3 N^{d-1}))$$

Where $\delta = d_B(\sigma_N, \frac{1}{|\mathcal{S}_N|})$ is the Bures distance from the max. mixed state.

Robustness against noise

Example (Robustness against depolarization ($d = 2$))

Provided that $\delta = d_B\left(\sigma_N, \frac{1}{|\mathcal{S}_N|}\right) \geq 1/N^{1/3}$

the QFI typically scales at least like $N^{4/3}$.

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Let F_k be the QFI after k particles were lost, then

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Caveat: If $k \propto N$, usefulness is lost, consistent with [15, 16].

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Can we access the “usefulness”?

Attaining the Heisenberg limit

Finding the right measurement can be tricky.

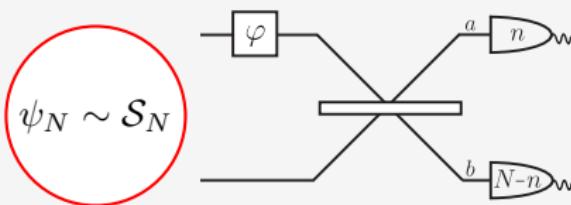
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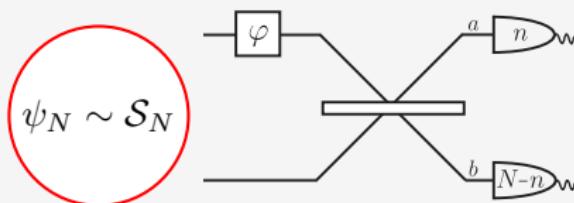


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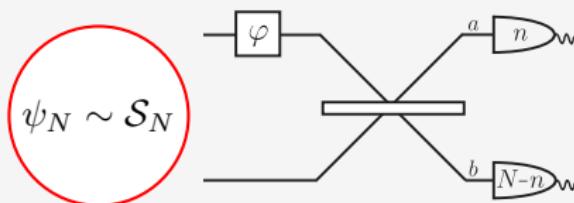
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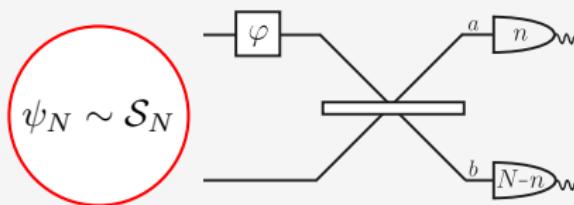
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Concretely: $\forall \varphi : \mathbb{E} F_{\text{cl}} \geq 0.0244 N^2$

How can we generate random symmetric states?

Generation with random circuits

- 💡 States drawn with a [unitary design](#) would be sufficient and they can be generated with [random circuits](#)!

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- Implies that polynomials (like the QFI) also are close
- [Measure concentration](#), albeit weaker, also carries over [20]

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! Results for qubits on **random circuits** efficiently yielding approximate **unitary designs** [18, 19] do **not carry over**.

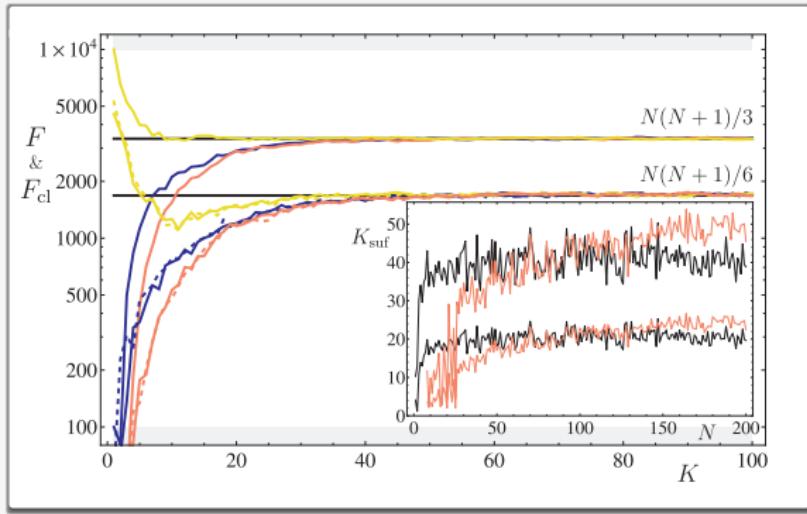
Don't know how K has to depend on N . \implies Numerics

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Numerics

$$N = 100, \quad |\psi_N\rangle = \sum_{n=0}^N \sqrt{x_n}|n, N-n\rangle$$



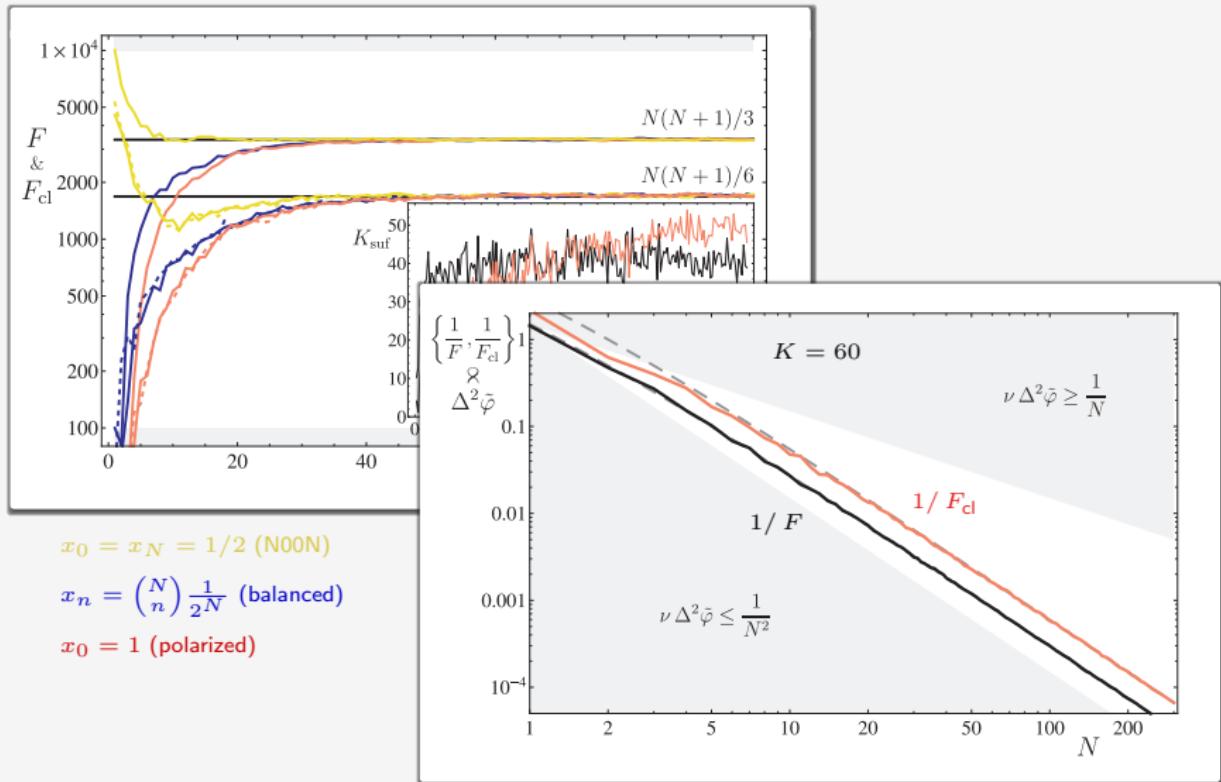
$$x_0 = x_N = 1/2 \text{ (N00N)}$$

$$x_n = \binom{N}{n} \frac{1}{2^N} \text{ (balanced)}$$

$$x_0 = 1 \text{ (polarized)}$$

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References

Thank you for your attention!

slides on www.cgogolin.de

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