Simulability of open quantum system dynamics
“A dissipative quantum Church-Turing theorem”

M. Kliesch, T. Barthel, C. Gogolin, M. Kastoryano, and J. Eisert

Dahlem Center for Complex Quantum Systems, Freie Universität Berlin

QCCC Workshop 2011
Simulability

What is (efficiently) simulatable?

Simulability

What is (efficiently) simulatable?

Here: Quantum many body dynamics

- On a quantum computer?
- On a classical computer?

1 Preliminaries

2 Trotterization of Liouvillian dynamics

3 Implications
   - Dissipative quantum computing
   - Efficient state preparation
   - Simulations on classical computers
   - Dissipative Church-Turing theorem
Preliminaries
Unitary vs. Liouvillian dynamics

Unitary:

equation of motion: \[ \frac{d}{dt} \rho(t) = -i[H, \rho(t)] \]
Unitary vs. Liouvillian dynamics

**Unitary:**

\[ \frac{d}{dt} \rho(t) = -i[H, \rho(t)] \]

**Liouvillian:**

\[ \frac{d}{dt} \rho(t) = \mathcal{L}(\rho(t)) \]
## Unitary vs. Liouvillian dynamics

<table>
<thead>
<tr>
<th></th>
<th>Unitary:</th>
<th>Liouvillian:</th>
</tr>
</thead>
<tbody>
<tr>
<td>equation of motion:</td>
<td>$\frac{d}{dt} \rho(t) = -i[H, \rho(t)]$</td>
<td>$\frac{d}{dt} \rho(t) = \mathcal{L}(\rho(t))$</td>
</tr>
<tr>
<td>time independent:</td>
<td>$\rho(t) = e^{-iHt} \rho(0) e^{iHt}$</td>
<td>$\rho(t) = e^{\mathcal{L}t} \rho(0)$</td>
</tr>
</tbody>
</table>
# Unitary vs. Liouvillian dynamics

<table>
<thead>
<tr>
<th></th>
<th>Unitary:</th>
<th>Liouvillian:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>equation of motion:</strong></td>
<td>$\frac{d}{dt}\rho(t) = -i[H, \rho(t)]$</td>
<td>$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t))$</td>
</tr>
<tr>
<td><strong>time independent:</strong></td>
<td>$\rho(t) = e^{-iHt}\rho(0)e^{iHt}$</td>
<td>$\rho(t) = e^{\mathcal{L}t}\rho(0)$</td>
</tr>
<tr>
<td><strong>time dependent:</strong></td>
<td>“time ordered product integrals”</td>
<td></td>
</tr>
</tbody>
</table>
Unitary vs. Liouvillian dynamics

**Equation of motion:**

- **Unitary:** \( \frac{d}{dt} \rho(t) = -i[H, \rho(t)] \)
- **Liouvillian:** \( \frac{d}{dt} \rho(t) = \mathcal{L}(\rho(t)) \)

**Time independent:**

- **Unitary:** \( \rho(t) = e^{-iHt} \rho(0) e^{iHt} \)
- **Liouvillian:** \( \rho(t) = e^{\mathcal{L}t} \rho(0) \)

**Time dependent:**

- **Unitary:**
  
- **Liouvillian:**
  
- "time ordered product integrals"

Propagator for \( t \geq s \geq 0 \)

\[
\rho(t) = T_{\mathcal{L}}(t, s)(\rho(s))
\]
Unitary vs. Liouvillian dynamics

**Equation of motion:**

Unitary: \( \frac{d}{dt} \rho(t) = -i[H, \rho(t)] \)

Liouvillian: \( \frac{d}{dt} \rho(t) = \mathcal{L}(\rho(t)) \)

**Time independent:**

Unitary: \( \rho(t) = e^{-iHt} \rho(0) e^{iHt} \)

Liouvillian: \( \rho(t) = e^{\mathcal{L}t} \rho(0) \)

**Time dependent:**

Unitary: None

Liouvillian: "time ordered product integrals"

**Propagator for** \( t \geq s \geq 0 \)

\( \rho(t) = T_{\mathcal{L}}(t, s)(\rho(s)) \)
Unitary vs. Liouvillian dynamics

**Equation of motion:**

**Unitary:** \[ \frac{d}{dt} \rho(t) = -i[H, \rho(t)] \]

**Liouvillian:** \[ \frac{d}{dt} \rho(t) = \mathcal{L}(\rho(t)) \]

**Time independent:**

**Unitary:** \[ \rho(t) = e^{-iHt} \rho(0) e^{iHt} \]

**Liouvillian:** \[ \rho(t) = e^{\mathcal{L}t} \rho(0) \]

**Time dependent:**

**Unitary:**

**Liouvillian:**

"time ordered product integrals"

Propagator for \( t \geq s \geq 0 \)

\[ \rho(t) = T_\mathcal{L}(t, s)(\rho(s)) \]
Distinguishability of propagators

Distinguishability of density matrices:

\[ \|\rho - \sigma\|_1 = \max_{0 \leq A \leq 1} \text{tr}(A(\rho - \sigma)) \]

Worst case estimate for propagators:

\[ \|T - T'\|_{1\to1} := \sup_{\|\rho\|_1 = 1} \|T(\rho) - T'(\rho)\|_1 \]
Trotterization of Liouvillian dynamics
$k$-local Liouvillian dynamics

\[ \mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_\Lambda \quad \mathcal{L}_\Lambda (\rho) = -i[H_\Lambda, \rho] + \sum_{\mu=1}^{d^k} 2L_{\Lambda,\mu} \rho L_{\Lambda,\mu}^\dagger - \{L_{\Lambda,\mu}^\dagger L_{\Lambda,\mu}, \rho\} \]
\( k \)-local Liouvillian dynamics

\[
\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_\Lambda \quad \mathcal{L}_\Lambda (\rho) = -i[H_\Lambda, \rho] + \sum_{\mu=1}^{d^k} 2L_{\Lambda, \mu} \rho L_{\Lambda, \mu}^\dagger - \{L_{\Lambda, \mu}^\dagger L_{\Lambda, \mu}, \rho\}
\]

Assumptions:

- \( N \) sites with finite local dimension \( d \)
**$k$-local Liouvillian dynamics**

\[
\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_\Lambda \quad \mathcal{L}_\Lambda(\rho) = -i[H_\Lambda, \rho] + \sum_{\mu=1}^{d^k} 2L_{\Lambda,\mu} \rho L_{\Lambda,\mu}^\dagger - \{L_{\Lambda,\mu}^\dagger L_{\Lambda,\mu}, \rho\}
\]

**Assumptions:**
- $N$ sites with finite local dimension $d$
- $k$-locality
$\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_{\Lambda} \quad \mathcal{L}_{\Lambda}(\rho) = -i[H_{\Lambda}, \rho] + \sum_{\mu=1}^{d^k} 2L_{\Lambda,\mu}\rho L_{\Lambda,\mu}^{\dagger} - \{L_{\Lambda,\mu}^{\dagger}L_{\Lambda,\mu}, \rho\}$

**Assumptions:**

- $N$ sites with finite local dimension $d$
- $k$-locality
\( k \)-local Liouvillian dynamics

\[
\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_\Lambda \quad \mathcal{L}_\Lambda(\rho) = -i[H_\Lambda, \rho] + \sum_{\mu = 1}^{d^k} 2L_{\Lambda, \mu}\rho L_{\Lambda, \mu}^\dagger - \{L_{\Lambda, \mu}^\dagger L_{\Lambda, \mu}, \rho\}
\]

Assumptions:
- \( N \) sites with finite local dimension \( d \)
- \( k \)-locality
\(k\)-local Liouvillian dynamics

\[
\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_\Lambda \quad \mathcal{L}_\Lambda(\rho) = -i[H_\Lambda, \rho] + \sum_{\mu=1}^{d^k} 2L_{\Lambda,\mu}\rho L_{\Lambda,\mu}^\dagger - \{L_{\Lambda,\mu}^\dagger L_{\Lambda,\mu}, \rho\}
\]

Assumptions:
- \(N\) sites with finite local dimension \(d\)
- \(k\)-locality
\(k\)-local Liouvillian dynamics

\[
\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_{\Lambda} \quad \mathcal{L}_{\Lambda}(\rho) = -i[H_{\Lambda}, \rho] + \sum_{\mu=1}^{d^{k}_{\Lambda}} 2L_{\Lambda,\mu}\rho L_{\Lambda,\mu}^{\dagger} - \{L_{\Lambda,\mu}^{\dagger}L_{\Lambda,\mu}, \rho\}
\]

Assumptions:

- \(N\) sites with finite local dimension \(d\)
- \(k\)-locality
- arbitrary time dependence
\[ \mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_\Lambda \quad \mathcal{L}_\Lambda(\rho) = -i[H_\Lambda, \rho] + \sum_{\mu=1}^{d^k} 2L_{\Lambda,\mu} \rho L_{\Lambda,\mu}^\dagger - \{L_{\Lambda,\mu}^\dagger L_{\Lambda,\mu}, \rho\} \]

**Assumptions:**

- \(N\) sites with finite local dimension \(d\)
- \(k\)-locality
- arbitrary time dependence
- \(\|H_\Lambda\|_\infty\) and \(\|L_{\Lambda,\mu}\|_\infty\) bounded independent of \(N\)
Trotterization – what it is and how we get there

What we are aiming for:

\[
T_L(\tau, 0) \approx \prod_{j=1}^{m} \prod_{\Lambda}^{K} T_{L\Lambda}(\Delta t_j, \Delta t(j-1))
\]

What we have to do:

1. Decompose \( T_L(\tau, 0) \) in time slices.
2. Approximate each time slice by applying local Liouvillians sequentially.
Trotterization of $\kappa$-local Liouvillian dynamics

\[ T_L(\tau, 0) \]
Trotterization of $\kappa$-local Liouvillian dynamics
Trotterization of $\kappa$-local Liouvillian dynamics
Trotterization of $k$-local Liouvilllan dynamics

\[
\Delta t \approx O(d^2 k K^2 \tau^2 \epsilon)
\]
Trotterization of $\kappa$-local Liouvillian dynamics
Trotterization of $k$-local Liouvillian dynamics
Trotterization of $\kappa$-local Liouvillian dynamics

\[ \Delta t \approx O(d^2 \kappa^2 \tau^2 \epsilon) \]
Trotterization of $\kappa$-local Liouvillian dynamics

\[ \approx \]

\[ m \]

\[ \vdots \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ \approx \]

\[ m \]

\[ \vdots \]

\[ 2 \]

\[ 1 \]
Trotterization of $k$-local Liouvillian dynamics

\[ \approx \]

\[ \Delta t^{1/2} \cdots m = O(d^2 k K^2 \tau^2 \epsilon) \]
Trotterization of $k$-local Liouvillian dynamics

\[ T_L(\tau, 0) \approx \]
Trotterization of $k$-local Liouvillian dynamics

$$T_{\mathcal{L}}(\tau, 0) \approx \sum_{j=1}^{m} \Delta t_j$$

$$m = \mathcal{O} \left( \frac{d^{2k} K^2 \tau^2}{\epsilon} \right)$$
A Trotter Formula for Liouvillian dynamics

Theorem 1 (Trotter decomposition [2])

Let $\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_{\Lambda}$ be a piecewise continuous time dependent, $k$-local Liouvillian acting on $N$ subsystems of dimension $d$, then

$$\left\| T_{\mathcal{L}}(\tau, 0) - \prod_{j=1}^{m} \prod_{\Lambda} T_{\mathcal{L}_{\Lambda}}(\Delta t_{j}, \Delta t_{(j-1)}) \right\|_{1 \rightarrow 1} \leq c K^{2} \tau \Delta t e^{b \Delta t},$$

with $b \in O(d^{k})$, $c \in O(d^{2k})$.

Theorem 1 (Trotter decomposition [2])

Let $\mathcal{L} = \sum_{\Lambda}^{K} \mathcal{L}_{\Lambda}$ be a piecewise continuous time dependent, $k$-local Liouvillian acting on $N$ subsystems of dimension $d$, then

$$\left\| T_{\mathcal{L}}(\tau, 0) - \prod_{j=1}^{m} \prod_{\Lambda} T_{\mathcal{L}_{\Lambda}^{\text{av}}}(\Delta t j, \Delta t (j - 1)) \right\|_{1 \rightarrow 1} \leq c K^{2} \tau \Delta t e^{b \Delta t},$$

with $b \in O(d^{k})$, $c \in O(d^{2k})$.

$$T_{\mathcal{L}_{\Lambda}^{\text{av}}}(\Delta t j, \Delta t (j - 1)) = \exp(\Delta t \mathcal{L}_{\Lambda}^{\text{av}}) \quad \mathcal{L}_{\Lambda}^{\text{av}} = \Delta t \int_{\Delta t (j - 1)}^{\Delta t j} \mathcal{L}_{\Lambda} dt$$

Stinespring and Solovay-Kitaev
Stinespring and Solovay-Kitaev

\[ T_{\mathcal{L}_1}(\Delta t, 0) \]
Stinespring and Solovay-Kitaev
Stinespring and Solovay-Kitaev

\[ T_{L_1}(\Delta t, 0) \]

Stinespring dilation

\[ \approx \]

Solovay-Kitaev
Implications
Implication 1

Power of dissipative quantum computing [3, 2]

Dissipative quantum computing with $k$-local, arbitrary time dependent Liouvillian dynamics is exactly as powerful as the circuit model.

Implication 2

**Limits on efficient state preparation [2]**

Even with arbitrary time-dependent $k$-local Liouvillian dynamics one can only reach **exponentially few states after polynomial time**.
Implication 2

**Limits on efficient state preparation [2]**

Even with arbitrary time-dependent $k$-local Liouvillian dynamics one can only reach exponentially few states after polynomial time.

$\epsilon$-net $\{\rho_i\}$:
Implication 2

Limits on efficient state preparation [2]

Even with arbitrary time-dependent $k$-local Liouvillian dynamics one can only reach exponentially few states after polynomial time.

$\varepsilon$-net $\{\rho_i\}$:

Smallest $\varepsilon$-nets:

$$\Omega \left( \exp(d^N) \right)$$

Implication 2

Limits on efficient state preparation [2]

Even with arbitrary time-dependent $k$-local Liouvillian dynamics one can only reach exponentially few states after polynomial time.

$\epsilon$-net $\{\rho_i\}$:

Smallest $\epsilon$-nets:

$$\Omega \left( \exp(d^N) \right)$$

Number of circuits for $\epsilon$-approximation:

$$O \left( \exp(N^{3k+2}\tau^4) \right)$$


Implication 3

Simulation on classical computers [2]

For fixed \( \tau \) dissipative dynamics can be simulated efficiently in \( N \) on classical computers.

Implication 3

Simulation on classical computers [2]

For fixed $\tau$ dissipative dynamics can be simulated efficiently in $N$ on classical computers.

Implication 3

Simulation on classical computers [2]

For fixed \( \tau \) dissipative dynamics can be simulated efficiently in \( N \) on classical computers.

local Observable

efficiently evaluable \( |\psi_0\rangle \)

Implication 4

Strong quantum Church-Turing thesis [2]

Every quantum mechanical process that can be thought of as a computation can be efficiently simulated in the unitary circuit model of quantum computing.

Implication 4

Strong quantum Church-Turing thesis [2]

Every quantum mechanical process that can be thought of as a computation can be efficiently simulated in the unitary circuit model of quantum computing.

Remember the assumption we made:

- \( N \) sites with finite local dimension \( d \)

Implication 4

Strong quantum Church-Turing thesis [2]

Every quantum mechanical process that can be thought of as a computation can be efficiently simulated in the unitary circuit model of quantum computing.

Remember the assumption we made:

- $N$ sites with finite local dimension $d$
- $k$-local Liouvillian dynamics

Implication 4

Strong quantum Church-Turing thesis [2]

Every quantum mechanical process that can be thought of as a computation can be efficiently simulated in the unitary circuit model of quantum computing.

Remember the assumption we made:

- $N$ sites with finite local dimension $d$
- $k$-local Liouvillian dynamics
- arbitrary time dependence

Implication 4

**Strong quantum Church-Turing thesis [2]**

Every quantum mechanical process that can be thought of as a computation can be efficiently simulated in the unitary circuit model of quantum computing.

Remember the assumption we made:
- $N$ sites with finite local dimension $d$
- $k$-local Liouvillian dynamics
- arbitrary time dependence

Arguably the most broadest setting that allows efficient simulation.

---

Summary

\[ T_L(\tau, 0) \approx \prod_{j=1}^{m} \prod_{\Lambda}^{K} T_{\mathcal{L}_\Lambda}(\Delta t j, \Delta t (j - 1)) \]

- \( k \)-local Liouvillian dynamics can be trotterized
Summary

\[ T_L(\tau, 0) \approx \prod_{j=1}^{m} \prod_{\Lambda}^K T_{\Lambda}(\Delta t j, \Delta t (j - 1)) \]

- $k$-local Liouvillian dynamics can be trotterized
- Dissipative quantum computing is no more powerful than the circuit model
Summary

\[ T_L(\tau, 0) \approx \prod_{j=1}^{m} \prod_{\Lambda}^{K} T_{\Lambda}(\Delta t_j, \Delta t(j - 1)) \]

- \( k \)-local Liouvillian dynamics can be \textbf{trotterized}
- Dissipative quantum computing is \textbf{no more powerful} than the circuit model
- Most states \textbf{can not be prepared efficiently}
$T_L(\tau, 0) \approx \prod_{j=1}^{m} \prod_{\Lambda} T_{\Lambda}(\Delta t j, \Delta t (j - 1))$

- $k$-local Liouvillian dynamics can be trotterized
- Dissipative quantum computing is no more powerful than the circuit model
- Most states can not be prepared efficiently
- $k$-local Liouvillian dynamics can be simulated classically (efficient in $N$, inefficient in $\tau$)
Summary

\[ T_L(\tau, 0) \approx \prod_{j=1}^{m} \prod_{\Lambda} T_{\Lambda}(\Delta t j, \Delta t (j - 1)) \]

- \( k \)-local Liouvillian dynamics can be trotterized
- Dissipative quantum computing is no more powerful than the circuit model
- Most states can not be prepared efficiently
- \( k \)-local Liouvillian dynamics can be simulated classically (efficient in \( N \), inefficient in \( \tau \))
- A strong quantum Church-Turing theorem holds
Collaborators

- Martin Kliesch
- Thomas Barthel
- Jens Eisert
- Michael Kastoryano
Thank you for your attention!

slides: www.cgogolin.de

References

[1] A. M. Turing,
“On computable numbers, with an application to the entscheidungsproblem”,

[2] M. Kliesch, T. Barthel, C. Gogolin, M. Kastoryano, and J. Eisert,
“Dissipative Quantum Church-Turing Theorem”,

“Quantum computation and quantum-state engineering driven by dissipation”,

[4] D. Poulin, A. Qarry, R. Somma, and F. Verstraete,
“Quantum Simulation of Time-Dependent Hamiltonians and the Convenient Illusion of Hilbert Space”,