Stability and efficient classical simulation of high temperature quantum states

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Investigating stability and simulatability of quantum states on lattice systems is a central topic in Hamiltonian complexity theory. We prove that above a universal critical temperature, only depending on local properties of the Hamiltonian, nearest-neighbor interactions are stable against distant Hamiltonian perturbations. As a consequence, local expectation values can be approximated in polynomial time. The stability theorem also provides a definition of temperature as a local quantity. The proof is based on novel exponential clustering of correlations result. We prove our clustering result via a reduction to a cluster expansion originally used to approximate thermal states by matrix-product operators.

Motivation
- Stability of thermal states against perturbations
- Does a Lieb-Robinson type quasi-locality hold in imaginary time?
- Physically speaking: On what length scale is temperature well-defined [1, 2, 3]?
- Stability ↔ correlations
- Behaviour of correlations
- Computational complexity ↔ thermal states

Main results

Theorem (Truncation formula)
Let \( J \leq E \) be a subset of edges, denote the corresponding interaction Hamiltonian by \( H_e \), and the interpolating Hamiltonian by \( H(s) = H(1-s) + s H_e \), with thermal state \( \rho = \text{e}^{-s H} \). Then, for any operator \( A \),
\[
\text{Tr}(A H_e(s)) - \text{Tr}(A \rho) = \beta \int_0^1 \text{d}s \text{cov}_s(\rho, A H_e(s)) \text{d}s.
\]

Theorem (Clustering of correlations at high temperatures)
Let the interaction (hyper)graph \( (V, E) \) have a growth constant \( \alpha \) and define the quantities
\[
\beta^* := \min \left( \frac{1}{1 + \sqrt{\gamma + 4/\gamma}}, \frac{2}{2 J} \right) \quad (\text{critical inverse temperature}),
\]
\[
\xi(J) := \left( \frac{1}{\ln(\alpha^{-1/2} \gamma^{1/2} - 1)} \right)^{-1} \quad (\text{correlation length}).
\]
Then, for every \( J < \beta^* \), parameter \( r \in [0, 1] \), and every two operators \( A \) and \( B \) with \( \text{d}(A, B) \geq L_1(\mathcal{A}, \mathcal{B}) \),
\[
\text{cov}_r(A, B) \leq \frac{4 L_1(\mathcal{A}, \mathcal{B})}{\ln(\beta^*/(\beta^* - r))} e^{-\xi(J) \text{d}(A,B)}.
\]

Implications
- Correlations measured by the averaged covariance are a measure of local stability against distant Hamiltonian perturbations.
- Temperature is intensive on a given length scale if and only if correlations are negligible (compared to \( 1/\xi \)) on that length scale.

Combined with the exponential clustering result:

Implication (Universal stability)
At high temperatures \( (J < \beta^*) \) thermal states are locally exponentially stable against distant Hamiltonian perturbations.

Corollary (Efficient classical local simulation)
Let \( |J| < \beta^* \), and let \( S \subset B \subset V \) be subsystems (see Figure 2) with \( d(S, \partial B) \geq L_1(\mathcal{S}, \partial B) \).
\[
\frac{\lambda(J)}{\lambda(\beta^*)} \leq \frac{1}{4} \left( 1 - e^{-\xi(J) \text{d}(S,\partial B)} \right).
\]

Ingredients for the clustering of correlations proof

Multiple swap-trick (4 topics)
With \( A^{-1} \triangleq A(1-1/A) \)
\[
\text{cov}_r(A, B) = \frac{1}{2} \sum_{w \in \mathcal{S}} \text{e}^{-\xi(J) \text{d}(S,\partial B)} \sum_{x \in \mathcal{W}} \text{e}^{-\xi(J) \text{d}(S,\partial B)}
\]

Effective Hamiltonian
\[
\tilde{H} = r H(1-r) + (1-r) H(1-r) \quad (\text{cluster expansion})
\]

Remark: Matrix product operators (MPOs)
- Local Hamiltonian is stable against distant Hamiltonian perturbations.
- For \( E \leq \frac{2}{3} \), Tensor size sub-exponentially large in the system size

Example (Critical temperature)

2D Ising Model (ferromagnetic, isotropic, without external field): 
- Our bound: \( 1/(\beta^* J) = 2/(\ln(1 + 1/(1 + 1/e) \approx 24.58 \) and
- Phase transition at: \( 1/(\beta^* J) = 2/(\ln(1 + \sqrt{2}) \approx 2.27 \) ... but remember, \( \beta^* \) provides a universal bound for all models.

References