# Constructing absolutely maximally entangled states and optimal quantum error correcting codes 

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${ }^{3}$ ICREA-Institució Catalana de Recerca i Estudis Avançats

$$
\begin{gathered}
\text { arXiv:1611.???? } \\
\langle\Psi| \text { QIS } 2016 \text { Barcelona } \\
\text { 2016-10-19 }
\end{gathered}
$$

## What are AME states?

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AME states
A state of $n$ particles is AME if for all $S \subset\{1, \ldots, n\}$

$$
|S| \leq\lfloor n / 2\rfloor \Longrightarrow \operatorname{Tr}_{S^{c}}|\psi\rangle\langle\psi| \propto \mathbb{1} .
$$

## Content of this talk


[1] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Życzkowski, Phys. Rev. A, 92 (3 2015), 032316
[2] W. Helwig and W. Cui, (2013), URL: https://arxiv.org/abs/1306.2536

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[^4]
## Classical error correcting codes

## Message

0

1

## Classical error correcting codes

Message Encoding

$0 \longrightarrow 000$
$1 \longrightarrow 111$

## Classical error correcting codes

Message Encoding Error


## Classical error correcting codes

Message Encoding Error Correction


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$$
\left[n=3, k=1, d_{H}=3\right]_{q=2}
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$$
\left\{\begin{array}{c}
0 \longrightarrow 000 \rightleftharpoons 010 \\
001 \\
000 \\
011 \\
1 \longrightarrow 111 \longrightarrow 110
\end{array}\right\} \longrightarrow 000
$$

## Maximal distance separable (MDS) codes

Is this optimal?

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$$
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## Yes!

Singleton bound [7]

$$
d_{H} \leq n-k+1
$$

[^5]
## Constructing linear MDS codes

Message Generator matrix Code word

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Message Generator matrix Code word
This only makes sense if you can take linear combinations of messages and code words!

Yes. Right. Solution: Finite fields
Integers modulo $q$ for $q$ prime are a finite field.

## Constructing linear MDS codes


$G$ Has standard form (by taking linear combinations of code words)

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G_{k \times n}=\left[\mathbb{1}_{k} \mid A\right]
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■ Smallest Hamming dist. given by smallest dist. to all zero code word $\Rightarrow$ Code is MDS iff any subset of $k$ columns of $G_{k \times n}$ is linearly independent $\Longleftrightarrow$ All square sub-matrices of $A$ are non-singular

## Constructing linear MDS codes with Singleton arrays

For $\gamma$ first primitive element of finite field define the Singleton array $[8,9]$ :

$$
S_{q}:=\begin{array}{ccccccc}
1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
1 & a_{1} & a_{2} & \ldots & a_{q-3} & a_{q-2} & \\
1 & a_{2} & a_{3} & \ldots & a_{q-2} & & \\
\vdots & \vdots & \vdots & . . & & & \\
1 & a_{q-3} & a_{q-2} & & & & \\
1 & a_{q-2} & & & & & w \\
1 & & & & & & w
\end{array}
$$

All square sub-matrices of $S_{q}$ are non-singular!

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A=\begin{array}{ccccc}
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1 & 1 & 1 & \ldots \\
1 & a_{1} & a_{2} & \ldots \\
1 & a_{2} & a_{3} & \ldots \\
\vdots & \vdots & \vdots & . . \\
S_{q}:= & \begin{array}{c}
1 \\
a_{q-3}
\end{array} & \begin{array}{c}
1 \\
a_{q-2} \\
a_{q-2}
\end{array} & \\
\begin{array}{l}
1 \\
a_{q-3} \\
1
\end{array} a_{q-2} & a_{q-2} & & \\
& & \\
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\end{array}
\end{array}
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[^9]
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■ Smallest Hamming distance between any two code words $d_{H}=n-k+1=\lceil n / 2\rceil+1$
■ Consider $\vec{v}, \vec{w} \in[q]^{\lfloor n / 2\rfloor}$, then the product states

$$
\begin{array}{ll} 
& \left|\vec{v} G_{\lfloor n / 2\rfloor \times n}\right\rangle \\
\text { and } & \left|\vec{w} G_{\lfloor n / 2\rfloor \times n}\right\rangle
\end{array}
$$

are orthogonal on all subsystems of size up to $\lceil n / 2\rceil+1$.

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■ Hence, we have an AME state!

$$
\langle\mathrm{QIS} \mid \Psi\rangle=\sum_{\vec{v} \in[q]\lfloor n / 2\rfloor}\left|\vec{v} G_{k \times n}\right\rangle
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## Minimal support AME states from MDS codes

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$$
|\Psi\rangle=\sum_{\vec{v} \in[q]\lfloor n / 2\rfloor}\left|\vec{v} G_{k \times n}\right\rangle
$$

## An example

Generator matrix of a $[6,3,4]_{5}$ MDS code:

$$
G_{3 \times 6}=\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 1 & 3 & 4
\end{array}\right]
$$

## An example

Generator matrix of a $[6,3,4]_{5}$ MDS code:

$$
G_{3 \times 6}=\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 1 & 3 & 4
\end{array}\right]
$$

Yields minimal support AME state for $n=6, q=5$ :

$$
|\Psi\rangle=\sum_{\vec{v} \in G F(5)^{3}}|\vec{v} G\rangle=\sum_{i, j, l=0}^{4}|i, j, l, i+j+l, i+2 j+3 l, i+3 j+4 l\rangle
$$

(All additions and multiplications modulo $q$.)
Can construct such states for all $n \leq q-1$ and $q$ prime.

## MDS codes from minimal support AME states

## MDS code



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## Quantum error correcting codes

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[[n, k, d]]_{q}
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## Message <br> Encoding

unitary

$$
|\psi\rangle \in\left(\mathbb{C}^{q}\right)^{\otimes k} \longrightarrow|\varphi\rangle \in \mathcal{C} \subset\left(\mathbb{C}^{q}\right)^{\otimes n}
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t \text { systems affected }
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\text { Message } & \text { Encoding } \\
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\text { measuring stabilizers } \\
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t \text { systems affected }
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$$

Quantum singleton bound [10-12]

$$
2 t+1=: d \leq \frac{n-k}{2}+1
$$

[^13]
## QECCs from minimal support AME states

## Conjecture

For every prime $q \geq n-1$ and $n$ a multiple of 4 there exists a $[[n, 1, n / 2-1]]_{q}$ QECC, whose code space $\mathcal{C}$ is spanned by AME states and we can construct it and its stabilizers explicitly for any $n$.

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We are $\varepsilon$-close to convincing ourselves that our proof strategy works. . .

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- Remember EPR state: $(U \otimes \mathbb{1})\left|\psi^{+}\right\rangle=\left(\mathbb{1} \otimes U^{\dagger}\right)\left|\psi^{+}\right\rangle$


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|\Psi\rangle=\sum_{j_{1} \ldots, j_{n}=1}^{q} c_{j_{1} \ldots, j_{n}}\left|j_{1} \ldots, j_{n}\right\rangle,
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$\operatorname{Tr}\left(\tilde{M}_{5}\right) \operatorname{Tr}\left(\tilde{M}_{6,7,8} \mathcal{E}\right)=$


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$$

$$
0=\operatorname{Tr}\left(\tilde{M}_{5}\right) \operatorname{Tr}\left(\tilde{M}_{6,7,8} \mathcal{E}\right)=
$$



## Summary



# 〈 xuantum 

the open journal for quantum science quantum-journal.org

## relevance visibility

 open data $\lambda$ o Uality social rigorous $\frac{\pi}{0}$ peer reviewed $\frac{\pi}{2}$ reproducible by researchers for researchers

## correctness

 theory transparent experiment online broad community essence © ${ }_{0}$ publagogic original

$$
\text { non-profit } \xlongequal{9}
$$

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## Thank you for your attention!

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## Ask Lídia and me about 〈 रuantum (poster in hall)!

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